

# LOGARITHMIC STABILITY IN DETERMINING TWO COEFFICIENTS IN A DISSIPATIVE WAVE EQUATION. EXTENSIONS TO CLAMPED EULER-BERNOULLI BEAM AND HEAT EQUATIONS

KAÏS AMMARI AND MOURAD CHOULLI

**ABSTRACT.** We are concerned with the inverse problem of determining both the potential and the damping coefficient in a dissipative wave equation from boundary measurements. We establish stability estimates of logarithmic type when the measurements are given by the operator who maps the initial condition to Neumann boundary trace of the solution of the corresponding initial-boundary value problem. We build a method combining an observability inequality together with a spectral decomposition. We also apply this method to a clamped Euler-Bernoulli beam equation. Finally, we indicate how the present approach can be adapted to a heat equation.

**Keywords:** Damping coefficient, potential, dissipative wave equation, boundary measurements, boundary observability, initial-to-boundary operator.

**MSC:** 93C25, 93B07, 93C20, 35R30.

## CONTENTS

1. Introduction	1
2. An abstract framework for the inverse source problem	4
3. Proof of Theorem 1.1	6
4. Stability around a non zero damping coefficient	7
5. An application to clamped Euler-Bernoulli beam	10
6. The case of a heat equation	12
Appendix A.	14
References	15

## 1. INTRODUCTION

We consider the following initial-boundary value problem (abbreviated to IBVP in the sequel) for the wave equation:

$$(1.1) \quad \begin{cases} \partial_t^2 u - \Delta u + q(x)u + a(x)\partial_t u = 0 & \text{in } Q = \Omega \times (0, \tau), \\ u = 0 & \text{on } \Sigma = \partial\Omega \times (0, \tau), \\ u(\cdot, 0) = u_0, \quad \partial_t u(\cdot, 0) = u_1, \end{cases}$$

where  $\Omega \subset \mathbb{R}^n$ ,  $n \geq 1$ , is a bounded domain with  $C^2$ -smooth boundary  $\partial\Omega$  and  $\tau > 0$ .

We assume in this text that the coefficients  $q$  and  $a$  are real-valued.

Under the assumption that  $q, a \in L^\infty(\Omega)$ , for each  $\tau > 0$  and  $\begin{pmatrix} u_0 \\ u_1 \end{pmatrix} \in H_0^1(\Omega) \times L^2(\Omega)$ , the IBVP (1.1) has a unique solution  $u_{q,a} \in C([0, \tau], H_0^1(\Omega))$  such that  $\partial_t u_{q,a} \in C([0, \tau], L^2(\Omega))$  (e.g. [7, pages 699-702]). On the other hand, by a classical energy estimate, we have

$$\|u_{q,a}\|_{C([0,\tau], H_0^1(\Omega))} + \|\partial_t u_{q,a}\|_{C([0,\tau], L^2(\Omega))} \leq C(\|u_0\|_{1,2} + \|u_1\|_0).$$

Here and henceforth,  $\|\cdot\|_p$  and  $\|\cdot\|_{s,p}$ ,  $1 \leq p \leq \infty$ ,  $s \in \mathbb{R}$ , denote respectively the usual  $L^p$ -norm and the  $W^{s,p}$ -norm.

We note that the constant  $C$  above is a non decreasing function of  $\|q\|_\infty + \|a\|_\infty$ .

Now, since  $u_{q,a}$  coincides with the solution of the IBVP (1.1) in which  $-q(x)u_{q,a} - a(x)\partial_t u_{q,a}$  is seen as a right-hand side, we can apply [17, Theorem 2.1] to get that  $\partial_\nu u_{q,a}$ , the derivative of the  $u_{q,a}$  in the direction of  $\nu$ , the unit outward normal vector to  $\partial\Omega$ , belongs to  $L^2(\Sigma)$ . Additionally, the mapping

$$\begin{pmatrix} u_0 \\ u_1 \end{pmatrix} \in H_0^1(\Omega) \times L^2(\Omega) \longrightarrow \partial_\nu u_{q,a} \in L^2(\Sigma)$$

defines a bounded operator.

Let  $\Gamma$  be a non empty open subset of  $\partial\Omega$  and  $\Upsilon = \Gamma \times (0, \tau)$ . To  $q, a \in L^\infty(\Omega)$ , we associate the initial-to-boundary (abbreviated to IB in the following) operator  $\Lambda_{q,a}$  defined by

$$\Lambda_{q,a} : \begin{pmatrix} u_0 \\ u_1 \end{pmatrix} \in H_0^1(\Omega) \times L^2(\Omega) \longrightarrow \partial_\nu u_{q,a}|_\Upsilon \in L^2(\Upsilon).$$

Clearly, from the preceding discussion,  $\Lambda_{q,a} \in \mathcal{B}(H_0^1(\Omega) \times L^2(\Omega), L^2(\Upsilon))$ .

We also consider two partial IB operators  $\Lambda_q$  and  $\tilde{\Lambda}_{q,a}$  which are given by

$$\Lambda_q(u_0) = \Lambda_{q,0} \begin{pmatrix} u_0 \\ 0 \end{pmatrix}, \quad \tilde{\Lambda}_{q,a}(u_1) = \Lambda_{q,a} \begin{pmatrix} 0 \\ u_1 \end{pmatrix}.$$

Therefore,  $\Lambda_q \in \mathcal{B}(H_0^1(\Omega), L^2(\Upsilon))$  and  $\tilde{\Lambda}_{q,a} \in \mathcal{B}(L^2(\Omega), L^2(\Upsilon))$ .

Next, we see that  $\partial_t u$  is the solution of the IBVP (1.1) corresponding to the initial conditions  $u_1$  and  $\Delta u_0 - qu_0 - au_1$ . Hence, repeating the preceding analysis with  $\partial_t u$  in place of  $u$ , we get

$$\Lambda_{q,a} \in \mathcal{B}([H_0^1(\Omega) \cap H^2(\Omega)] \times H_0^1(\Omega), H^1((0, \tau), L^2(\Gamma))).$$

Consequently,

$$\begin{aligned} \Lambda_q &\in \mathcal{B}(H_0^1(\Omega) \cap H^2(\Omega), H^1((0, \tau), L^2(\Gamma))), \\ \tilde{\Lambda}_{q,a} &\in \mathcal{B}(H_0^1(\Omega), H^1((0, \tau), L^2(\Gamma))). \end{aligned}$$

We are interested in the stability issue for the inverse problem consisting in the determination of both the potential  $q$  and the damping coefficient  $a$ , appearing in the IBVP (1.1), from the IB map  $\Lambda_{q,a}$ . We succeed in proving logarithmic stability estimates of determining  $q$  from  $\Lambda_q$ ,  $a$  from  $\tilde{\Lambda}_{q,a}$  and  $(q, a)$  from  $\Lambda_{q,a}$ .

We introduce the unbounded operators, defined on  $H_0^1(\Omega) \times L^2(\Omega)$ , as follows

$$\mathcal{A}_0 = \begin{pmatrix} 0 & I \\ \Delta & 0 \end{pmatrix}, \quad D(\mathcal{A}_0) = [H^2(\Omega) \cap H_0^1(\Omega)] \times H_0^1(\Omega)$$

and  $\mathcal{A} = \mathcal{A}_{q,a} = \mathcal{A}_0 + \mathcal{B}$  with  $D(\mathcal{A}) = D(\mathcal{A}_0)$ , where

$$\mathcal{B} = \mathcal{B}_{q,a} = \begin{pmatrix} 0 & 0 \\ -q & -a \end{pmatrix}.$$

Let

$$\mathcal{C} : D(\mathcal{A}_0) \rightarrow L^2(\Sigma) : \begin{pmatrix} \varphi \\ \psi \end{pmatrix} \longrightarrow \partial_\nu \varphi.$$

Since we deal with the wave equation, it is necessary to make assumptions on  $\Gamma$  and  $\tau$  in order to guarantee that our system is observable. To this end, we assume that  $\Gamma$  is chosen in such a way that there is  $\tau_0$  such that the pair  $(\mathcal{A}, \mathcal{C})$  is exactly observable for any  $\tau \geq \tau_0$ . We formulate the precise definition of exact observability in the next section in an abstract framework.

We give sufficient conditions ensuring that the pair  $(\mathcal{A}, \mathcal{C})$  is exactly observable. We fix  $x_0 \in \mathbb{R}^n \setminus \overline{\Omega}$  and we set

$$\Gamma_0 = \{x \in \partial\Omega; \nu(x) \cdot (x - x_0) > 0\} \quad \text{and} \quad d = \max_{x \in \overline{\Omega}} |x - x_0|.$$

Let us assume that  $\Gamma \supset \Gamma_0$ . Following [21, Theorem 7.2.3, page 233],  $(\mathcal{A}_0, \mathcal{C})$  is exactly observable with  $\tau \geq \tau_0 = 2d$ . In light of [21, Theorem 7.3.2, page 235] and the remark following it, we conclude that  $(\mathcal{A}, \mathcal{C})$  is also exactly observable for  $\tau \geq \tau_0$ , again with  $\tau_0 = 2d$ .

We mention that sharp sufficient conditions on  $\Gamma$  and  $\tau_0$  were given in a work by Bardos, Lebeau and Rauch [8].

Unless otherwise stated, for sake of simplicity, all operator norms will denoted by  $\|\cdot\|$ . Also,  $B_p$  (resp.  $B_{s,p}$ ) denote the unit ball of  $L^p(\Omega)$  (resp.  $W^{s,p}(\Omega)$ ).

We aim to prove in the present work the following theorem.

**Theorem 1.1.** *We assume that  $(\mathcal{A}_0, \mathcal{C})$  is exactly observable for  $\tau \geq \tau_0$ , for some  $\tau_0 > 0$ . Let  $0 \leq q_0 \in L^\infty(\Omega)$ , there is a constant  $\delta > 0$  so that*

$$(1.2) \quad \|q - q_0\|_2 \leq C \left| \ln \left( C^{-1} \|\Lambda_{q_0} - \Lambda_q\| \right) \right|^{-1/2}, \quad q \in q_0 + \delta B_{1,\infty},$$

and, for any  $m > 0$ ,

$$(1.3) \quad \|a\|_2 \leq C \left| \ln \left( C^{-1} \|\tilde{\Lambda}_{q_0,a} - \tilde{\Lambda}_{q_0,0}\| \right) \right|^{-1/2}, \quad a \in [\delta B_\infty] \cap [m B_{1,2}],$$

$$(1.4) \quad \|q - q_0\|_2 + \|a\|_0 \leq C \left| \ln \left( C^{-1} \|\Lambda_{q,a} - \Lambda_{q_0,0}\| \right) \right|^{-1/2}, \quad q \in q_0 + \delta B_{1,\infty}, \quad a \in [\delta B_\infty] \cap [m B_{1,2}].$$

Here,  $C$  is a generic constant not depending on  $q$  and  $a$ .

Theorem 1.1 gives only stability estimates at zero damping coefficient. The difficulty of stability estimates at a non zero damping coefficient is related to the fact that the operator  $\mathcal{A}$  is not necessarily diagonalizable. The main reason is that, contrary to case where  $a = 0$ , this operator is no longer skew-adjoint. We detail the stability estimate at a non zero damping coefficient in a separate section.

The problem of determining the potential in a wave equation from the so-called Dirichlet-to-Neumann (usually abbreviated to DN) map was initiated by Rakesh and Symes [18] (see also [10] and [13]). They prove that the potential can be recovered uniquely from the DN map provided that the length of the time interval is larger than the diameter of the space domain. The key point in their method is the construction of special solutions, called beam solutions. A sharp uniqueness result was proved by the so-called boundary control method. More details on this method can be found for instance in [6] and [16]. Also, Sun [20] establishes Hölder stability estimates and, most recently, Bao and Yun [4] improve the result of [20]. Specifically, they prove a nearly Lipschitz stability estimate. An extension was obtained by Bellassoued, Choulli and Yamamoto [3] in the case of a partial DN map by a method built on the quantification of the continuation of the solution of the wave equation from partial Cauchy data. We refer to the introduction of [3] for a short overview of inverse problems related to the wave equation. We finally quote a very recent paper by Bao and Zhang [5] dealing with sensitivity analysis of an inverse problem for the wave equation with caustics.

It is worthwhile to mention that contrary to hyperbolic inverse problems, for which the stability can be of Lipschitz, Hölder or logarithmic type, elliptic and parabolic inverse problems are always severely ill-posed. That is the corresponding stability estimates are in most cases of logarithmic type. In [1], Alessandrini gives an example in non destructive testing showing that the logarithmic stability is the best possible.

This text is organized as follows. We consider in Section 2 the inverse source problem for exactly observable systems in an abstract framework. This material is necessary to establish stability estimates for the determination of the potential and the damping coefficient appearing in the IBVP (1.1). We devote Section 3 to the proof of Theorem 1.1 and we give in Section 4 a sufficient condition which guarantees that  $\mathcal{A}$  is diagonalizable. The condition that  $\mathcal{A}$  is diagonalizable is used in an essential way to get a variant of Theorem 1.1 at a non zero damping coefficient. We apply in Section 5 our approach to a clamped Euler-Bernoulli beam equation. The possible adaptation of our method to a heat equation is discussed in Section 6. Due to the fact that a heat equation is not exactly observable but only observable at final time, we obtain a stability estimate only when we perturb the unknown coefficient by a finite dimensional subspace.

## 2. AN ABSTRACT FRAMEWORK FOR THE INVERSE SOURCE PROBLEM

Let  $H$  be a Hilbert space and  $A : D(A) \subset H \rightarrow H$  be the generator of continuous semigroup  $T(t)$ . An operator  $C \in \mathcal{B}(D(A), Y)$ ,  $Y$  is a Hilbert space which is identified with its dual space, is called an admissible observation for  $T(t)$  if for some (and hence for all)  $\tau > 0$ , the operator  $\Psi \in \mathcal{B}(D(A), L^2((0, \tau), Y))$  given by

$$(\Psi x)(t) = CT(t)x, \quad t \in [0, \tau], \quad x \in D(A),$$

has a bounded extension to  $H$ .

We introduce the notion of exact observability for the system

$$(2.1) \quad z'(t) = Az(t), \quad z(0) = x,$$

$$(2.2) \quad y(t) = Cz(t),$$

where  $C$  is an admissible observation for  $T(t)$ . Following the usual definition, the pair  $(A, C)$  is said exactly observable at time  $\tau > 0$  if there is a constant  $\kappa$  such that the solution  $(z, y)$  of (2.1) and (2.2) satisfies

$$\int_0^\tau \|y(t)\|_Y^2 dt \geq \kappa^2 \|x\|_H^2, \quad x \in D(A).$$

Or equivalently

$$(2.3) \quad \int_0^\tau \|(\Psi x)(t)\|_Y^2 dt \geq \kappa^2 \|x\|_H^2, \quad x \in D(A).$$

We consider the Cauchy problem

$$(2.4) \quad z'(t) = Az(t) + \lambda(t)x, \quad z(0) = 0,$$

and we set

$$(2.5) \quad y(t) = Cz(t), \quad t \in [0, \tau].$$

By Duhamel's formula, we have

$$(2.6) \quad y(t) = \int_0^t \lambda(t-s)CT(s)x ds = \int_0^t \lambda(t-s)(\Psi x)(s) ds.$$

Let

$$H_\ell^1((0, \tau), Y) = \{u \in H^1((0, \tau), Y); u(0) = 0\}.$$

We define the operator  $S : L^2((0, \tau), Y) \rightarrow H_\ell^1((0, \tau), Y)$  by

$$(2.7) \quad (Sh)(t) = \int_0^t \lambda(t-s)h(s) ds.$$

If  $E = S\Psi$ , then (2.6) takes the form

$$y(t) = (Ex)(t).$$

**Theorem 2.1.** *We assume that  $(A, C)$  is exactly observable for  $\tau \geq \tau_0$ , for some  $\tau_0 > 0$ . Let  $\lambda \in H^1((0, T))$  satisfies  $\lambda(0) \neq 0$ . Then  $E$  is one-to-one from  $H$  onto  $H_\ell^1((0, \tau), Y)$  and*

$$(2.8) \quad \frac{\kappa|\lambda(0)|}{\sqrt{2}} e^{-\tau \frac{\|\lambda'\|_{L^2((0, \tau))}^2}{|\lambda(0)|^2}} \|x\|_H \leq \|Ex\|_{H_\ell^1((0, \tau), Y)}, \quad x \in H.$$

*Proof.* First, taking the derivative with respect to  $t$  of each side of the integral equation

$$\int_0^t \lambda(t-s)\varphi(s) ds = \psi(t),$$

we get a Volterra equation of second kind

$$\lambda(0)\varphi(t) + \int_0^t \lambda'(t-s)\varphi(s) ds = \psi'(t).$$

Mimicking the proof of [15, Theorem 2, page 33], we obtain that this integral equation has a unique solution  $\varphi \in L^2((0, \tau), Y)$  and

$$\begin{aligned}\|\varphi\|_{L^2((0, \tau), Y)} &\leq C\|\psi'\|_{L^2((0, \tau), Y)} \\ &\leq C\|\psi\|_{H_\ell^1((0, \tau), Y)}.\end{aligned}$$

Here  $C = C(\lambda)$  is a constant.

Next, we estimate the constant  $C$  above. From the elementary convexity inequality  $(a + b)^2 \leq 2(a^2 + b^2)$ , we deduce

$$\|\lambda(0)|\varphi(t)\|_Y^2 \leq 2 \left( \int_0^t \frac{|\lambda'(t-s)|}{|\lambda(0)|} [|\lambda(0)|\|\varphi(s)\|_Y] ds \right)^2 + 2\|\psi'(t)\|_Y^2.$$

Thus,

$$|\lambda(0)|^2 \|\varphi(t)\|_Y^2 \leq 2 \frac{\|\lambda'\|_{L^2((0, \tau))}^2}{|\lambda(0)|^2} \int_0^t |\varphi(0)|^2 \|\varphi(s)\|_Y^2 ds + 2\|\psi'(t)\|_Y^2$$

by the Cauchy-Schwarz's inequality. Therefore, using Gronwall's lemma, we obtain in a straightforward manner that

$$\|\varphi\|_{L^2((0, \tau), Y)} \leq \frac{\sqrt{2}}{|\lambda(0)|} e^{\tau \frac{\|\lambda'\|_{L^2((0, \tau))}^2}{|\lambda(0)|^2}} \|\psi'\|_{L^2((0, \tau), Y)}$$

and then

$$\|\varphi\|_{L^2((0, \tau), Y)} \leq \frac{\sqrt{2}}{|\lambda(0)|} e^{\tau \frac{\|\lambda'\|_{L^2((0, \tau))}^2}{|\lambda(0)|^2}} \|S\varphi\|_{H_\ell^1((0, \tau), Y)}.$$

In light of (2.3), we end up getting

$$\|Ex\|_{H_\ell^1((0, \tau), Y)} \geq \frac{\kappa|\lambda(0)|}{\sqrt{2}} e^{-\tau \frac{\|\lambda'\|_{L^2((0, \tau))}^2}{|\lambda(0)|^2}} \|x\|_H.$$

□

We shall need a variant of Theorem 2.1. If  $(A, C)$  is as in Theorem 2.1, then it follows from [21, Proposition 6.3.3, page 189] that there is  $\delta > 0$  such that for any  $P \in \mathcal{B}(H)$  satisfying  $\|P\| \leq \delta$ ,  $(A + P, C)$  is exactly observable with  $\kappa(P + A) \geq \kappa/2$ .

We define  $E^P$  similarly to  $E$  by replacing  $A$  by  $A + P$ .

**Theorem 2.2.** *We assume that  $(A, C)$  is exactly observable for  $\tau \geq \tau_0$ , for some  $\tau_0 > 0$ . Let  $\lambda \in H^1((0, T))$  satisfies  $\lambda(0) \neq 0$ . There is  $\delta > 0$  such that, for any  $P \in \mathcal{B}(H)$  satisfying  $\|P\| \leq \delta$ ,  $E^P$  is one-to-one from  $H$  onto  $H_\ell^1((0, \tau), Y)$  and*

$$(2.9) \quad \frac{\kappa|\lambda(0)|}{2\sqrt{2}} e^{-\tau \frac{\|\lambda'\|_{L^2((0, \tau))}^2}{|\lambda(0)|^2}} \|x\|_H \leq \|E^P x\|_{H_\ell^1((0, \tau), Y)}, \quad x \in H.$$

We now apply the preceding theorem to the following IBVP for the wave equation

$$(2.10) \quad \begin{cases} \partial_t^2 u - \Delta u + q(x)u + a(x)\partial_t u = \lambda(t)f(x) & \text{in } Q, \\ u = 0 & \text{on } \Sigma, \\ u(\cdot, 0) = 0, \quad \partial_t u(\cdot, 0) = 0. \end{cases}$$

We recall that

$$\mathcal{A}_0 = \begin{pmatrix} 0 & I \\ \Delta & 0 \end{pmatrix}, \quad D(\mathcal{A}_0) = [H^2(\Omega) \cap H_0^1(\Omega)] \times H_0^1(\Omega)$$

and  $\mathcal{A} = \mathcal{A}_{q,a} = \mathcal{A}_0 + \mathcal{B}_{q,a}$  with  $D(\mathcal{A}) = D(\mathcal{A}_0)$ , where

$$\mathcal{B}_{q,a} = \begin{pmatrix} 0 & 0 \\ -q & -a \end{pmatrix}.$$

Also

$$\mathcal{C} : D(\mathcal{A}_0) \rightarrow L^2(\Gamma) : \begin{pmatrix} \varphi \\ \psi \end{pmatrix} \longrightarrow \partial_\nu \varphi.$$

We fix  $q_0, a_0 \in L^\infty(\Omega)$  and we assume that  $(\mathcal{A}_{q_0, a_0}, \mathcal{C})$  is exactly observable with constant  $\kappa$ . This is the case when  $\Gamma \supset \Gamma_0 = \{x \in \partial\Omega; \nu(x) \cdot (x - x_0) > 0\}$  (see for instance [12, Theorem 1.2, page 141]<sup>1</sup>).

**Corollary 2.1.** *There is  $\delta > 0$  such that, for any  $q \in q_0 + \delta B_{1,\infty}$  and  $a \in a_0 + \delta B_\infty$ , we have*

$$\|f\|_2 \leq \frac{2\sqrt{2}}{\kappa|\lambda(0)|} e^{\tau \frac{\|\lambda'\|_{L^2((0,\tau))}^2}{|\lambda(0)|^2}} \|\partial_\nu u_f\|_{H^1((0,\tau), L^2(\Gamma))},$$

where  $u_f$  is the solution of the IBVP (2.10).

This is nothing else but a Lipschitz stability estimate for the inverse problem of determining the source term  $f$  from the boundary data  $\partial_\nu u_f|_\Gamma$ , when  $\lambda$  is supposed to be known.

### 3. PROOF OF THEOREM 1.1

*Proof of Theorem 1.1.* Let  $(\lambda_k)$  and  $(\phi_k)$  be respectively the sequence of Dirichlet eigenvalues of  $-\Delta + q_0$ , counted according to their multiplicity, and the corresponding eigenvectors. We assume that the sequence  $(\phi_k)$  forms an orthonormal basis of  $L^2(\Omega)$ .

We recall that according to the min-max principle, the following two-sided estimates hold

$$(3.1) \quad c^{-1}k^{2/n} \leq \lambda_k \leq ck^{2/n}.$$

Here, the constant  $c > 1$  depends only on  $\Omega$  and  $q_0$ .

Let  $u_q$  be the solution of the IBVP (1.1) corresponding to  $q$ ,  $a = 0$ ,  $u_0 = \phi_k$  and  $u_1 = 0$ . Taking into account that  $u_{q_0} = \cos(t\sqrt{\lambda_k})\phi_k$  is the solution of the IBVP (1.1) corresponding to  $q = q_0$ ,  $a = 0$ ,  $u_0 = \phi_k$  and  $u_1 = 0$ , we see that  $u = u_q - u_{q_0}$  is the solution of the IBVP

$$(3.2) \quad \begin{cases} \partial_t^2 u - \Delta u + qu = -(q - q_0) \cos(t\sqrt{\lambda_k})\phi_k & \text{in } Q, \\ u = 0 & \text{on } \Sigma, \\ u(\cdot, 0) = 0, \partial_t u(\cdot, 0) = 0. \end{cases}$$

In the remaining part of this proof,  $C$  is a generic constant independent on  $k$ .

Let  $\delta$  be as in Corollary 2.1. If  $q \in q_0 + \delta B_{1,\infty}$ , we get by applying Corollary 2.1

$$\|(q - q_0)\phi_k\|_2 \leq Ce^{C\lambda_k} \|\partial_\nu u\|_{H^1((0,\tau), L^2(\Gamma))}.$$

Since  $|(q - q_0, \phi_k)| \leq |\Omega|^{1/2} \|(q - q_0)\phi_k\|_{L^2(\Omega)}$  by Cauchy-Schwarz's inequality, the last inequality entails

$$|(q - q_0, \phi_k)| \leq Ce^{C\lambda_k} \|\partial_\nu u\|_{H^1((0,\tau), L^2(\Gamma))}.$$

But,  $\partial_\nu u = (\Lambda_{q_0} - \Lambda_q)\phi_k$ . Therefore

$$(3.3) \quad |(q - q_0, \phi_k)|^2 \leq Ce^{C\lambda_k} \|\Lambda_{q_0} - \Lambda_q\|^2.$$

Let  $\lambda \geq \lambda_1$  and  $N = N(\lambda)$  be the smallest integer so that  $\lambda_N \leq \lambda < \lambda_{N+1}$ . Then

$$\begin{aligned} \|q - q_0\|_2^2 &= \sum_k |(q - q_0, \phi_k)|^2 \\ &= \sum_{k \leq N} |(q - q_0, \phi_k)|^2 + \sum_{k > N} |(q - q_0, \phi_k)|^2 \\ &\leq \sum_{k \leq N} |(q - q_0, \phi_k)|^2 + \frac{1}{\lambda} \sum_{k > N} \lambda_k |(q - q_0, \phi_k)|^2 \\ &\leq \sum_{k \leq N} |(q - q_0, \phi_k)|^2 + \frac{C\delta^2}{\lambda}. \end{aligned}$$

---

<sup>1</sup>We note that from the proof of this theorem it is not possible to extract the dependance of  $\kappa$  on  $q_0$  and  $a_0$ .

Here we used the fact that  $\left(\sum_{k \geq 1} (1 + \lambda_k)(\cdot, \varphi_k)^2\right)^{1/2}$  defines an equivalent norm on  $H^1(\Omega)$ .

In light of (3.3), we get

$$\|q - q_0\|_2^2 \leq CNe^{C\lambda} \|\Lambda_{q_0} - \Lambda_q\|^2 + \frac{C\delta^2}{\lambda}.$$

By (3.1),  $N \leq C\lambda^{n/2}$ . Hence

$$\|q - q_0\|_2^2 \leq Ce^{C\lambda} \|\Lambda_{q_0} - \Lambda_q\|^2 + \frac{C\delta^2}{\lambda}.$$

Minimizing with respect to  $\lambda$ , we obtain that there is  $\delta_0 > 0$  such that if  $\|\Lambda_{q_0} - \Lambda_q\| \leq \delta_0$ , then

$$\|q - q_0\|_2 \leq C |\ln(C^{-1} \|\Lambda_{q_0} - \Lambda_q\|)|^{-1/2}.$$

Estimate (1.2) follows then from the continuity of the mapping

$$q \in L^\infty(\Omega) \rightarrow \Lambda_q \in \mathcal{B}(H_0^1(\Omega) \cap H^2(\Omega), H^1((0, \tau), L^2(\Gamma))).$$

We proceed similarly for proving (1.3). In the actual case we have to replace the previous  $u_{q_0}$  by  $u_{q_0} = \lambda_k^{-1} \sin(t\sqrt{\lambda_k})\phi_k$ , corresponding to the initial conditions  $u_0 = 0$  and  $u_1 = \phi_k$ . Therefore, we have in place of (3.2)

$$(3.4) \quad \begin{cases} \partial_t^2 u - \Delta u + a\partial_t u = -a \cos(t\sqrt{\lambda_k})\phi_k & \text{in } Q, \\ u = 0 & \text{on } \Sigma, \\ u(\cdot, 0) = 0, \partial_t u(\cdot, 0) = 0. \end{cases}$$

We continue as in the preceding case by establishing the estimate

$$|(a, \phi_k)|^2 \leq Ce^{C\lambda_k} \|\tilde{\Lambda}_{q_0, a} - \tilde{\Lambda}_{q_0, 0}\|$$

and we complete the proof of (1.3) as above.

We end the proof by showing how we proceed for proving (1.4). Taking into account that the solution corresponding to  $q = q_0$ ,  $a = 0$ ,  $u_0 = \phi_k$  and  $u_1 = i\lambda_k \phi_k$  is  $u_{q_0} = e^{i\sqrt{\lambda_k}t}\phi_k$ , then in place of (3.2) we have the following IBVP

$$(3.5) \quad \begin{cases} \partial_t^2 u - \Delta u + qu + a\partial_t u = -[(q - q_0) + i\sqrt{\lambda_k}a]e^{i\sqrt{\lambda_k}t}\phi_k, & \text{in } Q, \\ u = 0 & \text{on } \Sigma, \\ u(\cdot, 0) = 0, \partial_t u(\cdot, 0) = 0. \end{cases}$$

We can argue one more time as in the proof of (1.2). We find

$$|(\varphi, q - q_0) + i\sqrt{\lambda_k}(\varphi, a)|^2 \leq Ce^{C\lambda_k} \|\Lambda_{q, a} - \Lambda_{q_0, 0}\|^2,$$

entailing

$$\begin{aligned} |(\varphi, q - q_0)|^2 &\leq Ce^{C\lambda_k} \|\Lambda_{q, a} - \Lambda_{q_0, 0}\|^2, \\ |(\varphi, a)|^2 &\leq Ce^{C\lambda_k} \|\Lambda_{q, a} - \Lambda_{q_0, 0}\|^2. \end{aligned}$$

We end up getting (1.4) by mimicking the rest of the proof of estimate (1.2).  $\square$

#### 4. STABILITY AROUND A NON ZERO DAMPING COEFFICIENT

We limit ourselves to the one dimensional case and, for sake of simplicity, we take  $q$  identically equal to zero. But the analysis we carry out in the present section is still applicable for any non negative bounded potential.

We assume in the present section that  $\Omega = (0, \pi)$ . We introduced in the first section the unbounded operators, defined on  $\mathcal{H} = H_0^1(\Omega) \times L^2(\Omega)$ ,

$$\mathcal{A}_0 = \begin{pmatrix} 0 & I \\ \frac{d^2}{dx^2} & 0 \end{pmatrix}, \quad D(\mathcal{A}_0) = [H^2(\Omega) \cap H_0^1(\Omega)] \times H_0^1(\Omega) := \mathcal{H}_1$$

and  $\mathcal{A}_a = \mathcal{A}_0 + \mathcal{B}_a$  with  $D(\mathcal{A}) = D(\mathcal{A}_0)$ , where

$$\mathcal{B}_a = \begin{pmatrix} 0 & 0 \\ 0 & -a \end{pmatrix}.$$

From [21, Proposition 6.2.1, page 180],  $(\mathcal{A}_0, \mathcal{C})$  is exactly observable for any  $\tau \geq 2\pi$  when

$$\mathcal{C} : \begin{pmatrix} \varphi \\ \psi \end{pmatrix} \in D(\mathcal{A}_0) \longrightarrow \frac{d\varphi}{dx}(0).$$

On the other hand, it follows from [21, Proposition 3.7.7, page 101] that the skew-adjoint operator  $\mathcal{A}_0$  is diagonalizable with eigenvalues  $\lambda_k = ik$ ,  $k \in \mathbb{Z}^*$ , corresponding to the orthonormal basis  $(g_k)$ , where

$$g_k = \frac{1}{\sqrt{2}} \begin{pmatrix} \frac{f_k}{ik} \\ f_k \end{pmatrix}, \quad k \in \mathbb{Z}^*,$$

where  $(f_k)_{k \in \mathbb{N}}$  is an orthonormal basis of  $L^2(\Omega)$  consisting of eigenfunctions of the unbounded operator  $A_0 = \frac{d^2}{dx^2}$  under Dirichlet boundary condition and  $f_{-k} = -f_k$ ,  $k \in \mathbb{N}^*$ .

Let  $\mathcal{H}_\pm$  be the closure in  $\mathcal{H}$  of  $\text{span}\{g_{\pm k}; k \in \mathbb{N}^*\}$ . Clearly,  $\mathcal{H} = \mathcal{H}_+ \oplus \mathcal{H}_-$  and  $\mathcal{H}_\pm$  is invariant under  $\mathcal{A}_0$ . Let then  $\mathcal{A}_0^\pm : \mathcal{H}_\pm \rightarrow \mathcal{H}_\pm$  be the unbounded operator given by  $\mathcal{A}_0^\pm = \mathcal{A}_0|_{\mathcal{H}_\pm}$  and

$$D(\mathcal{A}_0^\pm) = \{u \in \mathcal{H}_\pm; \sum_{k \in \mathbb{N}^*} k^2 |\langle u, g_{\pm k} \rangle|^2 < \infty\}.$$

Here  $\langle \cdot, \cdot \rangle$  is the scalar product in  $\mathcal{H}$ .

Let  $\mathcal{A}_{a_0}^\pm = \mathcal{A}_0^\pm + \mathcal{B}_{a_0}$  and set

$$\varrho = \sum_{k \geq 1} \frac{1}{(2k+1)^2} \quad \text{and} \quad \alpha = \frac{1}{2\sqrt{2(1+\varrho)}}.$$

In light of [19, Theorem 2 and Lemma 10], we get

**Theorem 4.1.** *Under the assumption*

$$\rho := \|a_0\|_\infty < \alpha,$$

*the spectrum of  $\pm \mathcal{A}_{a_0}^\pm$  consists in a sequence  $(i\mu_k^\pm)$  such that, for any  $\delta \in (0, 1 - \rho^2/\alpha^2)$ , there is an integer  $\tilde{k}$  such that*

$$|i\mu_k^\pm - ik| \leq \bar{\alpha} = \bar{\alpha}(a_0) := \frac{\rho}{\sqrt{4\rho^2 + \delta}}, \quad k \geq \tilde{k}.$$

*In addition,  $\mathcal{H}_\pm$  admits a Riesz basis  $(\phi_k^\pm) = \left( \begin{pmatrix} \varphi_k^\pm \\ i\mu_k^\pm \varphi_k^\pm \end{pmatrix} \right)_{k \in \mathbb{N}^*}$ , each  $\phi_k^\pm$  is an eigenfunction corresponding to  $i\mu_k^\pm$ .*

We denote by  $(\tilde{\phi}_k^\pm)$  the Riesz basis biorthogonal to  $(\phi_k^\pm)$  and define the sequence  $(\phi_k)_{k \in \mathbb{Z}^*}$  (resp.  $(\tilde{\phi}_k)_{k \in \mathbb{Z}^*}$ ) as follows  $\phi_{-k} = -\phi_k^-$  and  $\phi_k = \phi_k^+$  (resp.  $\tilde{\phi}_{-k} = -\tilde{\phi}_k^-$  and  $\tilde{\phi}_k = \tilde{\phi}_k^+$ ),  $k \in \mathbb{N}^*$ . Set also  $\mu_{-k} = -\mu_k^-$  and  $\mu_k = \mu_k^+$ ,  $k \in \mathbb{N}^*$ . Therefore,  $\mathcal{A}_{a_0} \phi_k = i\mu_k \phi_k$ ,  $k \in \mathbb{Z}^*$ , and, for any  $u \in \mathcal{H}$ ,

$$u = \sum_{k \in \mathbb{Z}} \langle u, \tilde{\phi}_k \rangle \phi_k = \sum_{k \in \mathbb{Z}} \langle u, \phi_k \rangle \tilde{\phi}_k.$$

Additionally,

$$(4.1) \quad \alpha \|u\|_{\mathcal{H}}^2 \leq \sum_{k \in \mathbb{Z}^*} |\langle u, \tilde{\phi}_k \rangle|^2, \quad \sum_{k \in \mathbb{Z}^*} |\langle u, \phi_k \rangle|^2 \leq \beta \|u\|_{\mathcal{H}}^2,$$

where the constants  $\alpha$  and  $\beta$  do not depend on  $u$  (see for instance [21, Lemma 252, page 37]).

We pick  $a_0$  as in the preceding theorem. Then it is straightforward to check that  $u_{a_0} = e^{i\mu_k t} \varphi_k$ ,  $k \in \mathbb{Z}^*$ , is the solution of the IBVP (1.1) with  $q = 0$ ,  $a = a_0$ ,  $\begin{pmatrix} u_0 \\ u_1 \end{pmatrix} = \phi_k$ . If  $u_a$  is the solution of the IBVP (1.1), then  $u = u_a - u_{a_0}$  is the solution of the IBVP



$$(4.2) \quad \begin{cases} \partial_t^2 u - \Delta u + a(x) \partial_t u = (a_0 - a) i \mu_k e^{i \mu_k t} \varphi_k & \text{in } Q, \\ u = 0 & \text{on } \Sigma, \\ u(\cdot, 0) = 0, \partial_t u(\cdot, 0) = 0. \end{cases}$$

We fixe  $\delta$  as in the statement of Theorem 4.1. Then, for some integer  $\tilde{k}$ ,

$$\begin{aligned} |e^{i \mu_k t}| &\leq e^{|i \mu_k - i k| |t|} |e^{i k t}| \leq e^{\overline{\alpha} \tau}, \quad |k| \geq \tilde{k}, \\ |\mu_k| &\leq |k| + \overline{\alpha}, \quad |k| \geq \tilde{k}. \end{aligned}$$

These estimates at hand, we can proceed as in the previous section to get, where  $\psi_k = i \mu_k \varphi_k$ ,

$$(4.3) \quad |(a - a_0, \psi_k)|^2 = \left| \left\langle \begin{pmatrix} 0 \\ a - a_0 \end{pmatrix}, \phi_k \right\rangle \right|^2 \leq C e^{C k^2} \|\Lambda_a - \Lambda_{a_0}\|^2.$$

It follows from (4.1),

$$(4.4) \quad \alpha \|a - a_0\|_2^2 = \alpha \left\| \begin{pmatrix} 0 \\ a - a_0 \end{pmatrix} \right\|_{\mathcal{H}}^2 \leq \sum_{|k| \geq 1} \left| \left\langle \begin{pmatrix} 0 \\ a - a_0 \end{pmatrix}, \phi_k \right\rangle \right|^2.$$

In light of (4.3) and (4.4), we have

$$(4.5) \quad \begin{aligned} \alpha \|a - a_0\|_2^2 &\leq C N e^{C \lambda} \|\Lambda_a - \Lambda_{a_0}\|^2 + \frac{1}{\lambda} \sum_{|k| > N} k^2 |(a - a_0, \psi_k)|^2 \\ &\leq C N e^{C \lambda} \|\Lambda_a - \Lambda_{a_0}\|^2 + \frac{1}{\lambda} \sum_{|k| \geq 1} k^2 |(a - a_0, \psi_k)|^2. \end{aligned}$$

Here  $\lambda \geq \lambda_1$  and  $N = N(\lambda)$  be the smallest integer satisfying  $N^2 \leq \lambda < (N+1)^2$ .

We note that we cannot pursue the proof similarly to that of (1.2) because  $(\psi_k)$  is not necessarily an orthonormal basis of  $L^2(\Omega)$ . So instead of the boundedness of  $a - a_0$  in  $H^1(\Omega)$ , we make the assumption, where  $m > 0$  is fixed,

$$(4.6) \quad \sum_{|k| \geq 1} k^2 |(a - a_0, \psi_k)|^2 \leq m.$$

Under the assumption (4.6), (4.5) entails

$$\alpha \|a - a_0\|_2^2 \leq C e^{C \lambda} \|\tilde{\Lambda}_a - \tilde{\Lambda}_{a_0}\|^2 + \frac{m}{\lambda}.$$

where  $\tilde{\Lambda}_a = \tilde{\Lambda}_{0,a}$  and  $\tilde{\Lambda}_{a_0} = \tilde{\Lambda}_{0,a_0}$ .

The same minimization argument used in the proof of (1.2) (see Section 3) allows us to prove the following theorem.

**Theorem 4.2.** *There exist two constants  $C > 0$  and  $\delta > 0$  so that*

$$\|a - a_0\|_2 \leq C \left| \ln \left( C^{-1} \|\tilde{\Lambda}_a - \tilde{\Lambda}_{a_0}\| \right) \right|^{-1/2}, \quad a \in a_0 + \delta B_\infty \text{ and (4.6) holds.}$$

*Remark 4.1.* Let us explain briefly why the result of this section can not be extended to a higher dimensional case. The main reason is that, even for simple geometries, the eigenvalues of the unperturbed operators  $\mathcal{A}_0^\pm$  do not satisfy a gap condition which is the main assumption in [19, Theorem 2]. If  $(\rho_k)$ ,  $\rho_k = k$ , is the sequence of eigenvalues of  $\pm \mathcal{A}_0^\pm$ , we used in an essential way that

$$\rho_{k+1} - \rho_k = 1.$$

When  $\Omega = (0, a) \times (0, b)$ , the eigenvalues operator  $\mathcal{A}_0^+$  consist in the sequence  $\left( \pi^2 \left( \frac{k^2}{a^2} + \frac{\ell^2}{b^2} \right) \right)_{k, \ell \in \mathbb{N}^*}$ . These eigenvalues are simple when  $\frac{a^2}{b^2} \notin \mathbb{Q}$  but can condensate in finite interval and therefore they don't satisfy a gap condition like in the one dimensional case.

## 5. AN APPLICATION TO CLAMPED EULER-BERNOULLI BEAM

For the same reason as in the preceding section, we limit our analysis to the one dimensional case. So we let  $\Omega = (0, 1)$ .

We introduce the following spaces

$$\begin{aligned} H_0 &= L^2(0, 1), \\ H_{1/2} &= H_0^2(\Omega), \\ H_1 &= H^4(0, 1) \cap H_0^2(\Omega). \end{aligned}$$

The natural norm of  $H_s$  will denoted by  $\|\cdot\|_s$ ,  $s \in \{0, 1/2, 1\}$ .

On  $\mathcal{H} = H_{1/2} \times H_0$ , we introduce the unbounded operator  $\mathcal{A}$  given by

$$\mathcal{A} = \begin{pmatrix} 0 & I \\ -\frac{d^4}{dx^4} & 0 \end{pmatrix}, \quad D(\mathcal{A}) = H_1 \times H_{1/2} := \mathcal{H}_1.$$

We consider a torque observation at an end point. We define then  $C : \mathcal{H}_1 \rightarrow \mathbb{C}$  by

$$C \begin{pmatrix} \varphi \\ \psi \end{pmatrix} = \frac{d^2 \varphi}{dx^2}(0).$$

We are concerned with following IBVP for the clamped Euler-Bernoulli beam equation

$$(5.1) \quad \begin{cases} \partial_t^2 u + \partial_x^4 u = 0 & \text{in } Q, \\ u(0, \cdot) = u(1, \cdot) = 0 & \text{on } (0, \tau), \\ \partial_x u(0, \cdot) = \partial_x u(1, \cdot) = 0 & \text{on } (0, \tau), \\ u(\cdot, 0) = u_0, \partial_t u(\cdot, 0) = u_1. \end{cases}$$

From [21, Proposition 3.7.6, page 100],  $\mathcal{A}$  is skew-adjoint and therefore it generates a unitary group on  $\mathcal{H}$ . Consequently, for any  $\begin{pmatrix} u_0 \\ u_1 \end{pmatrix} \in \mathcal{H}_1$  the IBVP (5.1) has a unique solution  $u$  so that  $(u, u') \in C([0, \tau], \mathcal{H}_1) \cap C^1([0, \tau], \mathcal{H})$ . Moreover, by [21, Proposition 6.10.1, page 270],  $(\mathcal{A}, C)$  is exactly observable for any  $\tau > 0$  and there is a constant  $\kappa$  such that

$$(5.2) \quad \kappa^2(\|u_0\|_{1/2}^2 + \|u_1\|_0^2) \leq \|\partial_x^2 u(0, \cdot)\|_{L^2((0, \tau))}^2.$$

Here the constant  $\kappa$  is independent on  $u_0$  and  $u_1$ .

Let  $\mathcal{B}_a$  be the operator, where  $a = a(x)$ ,

$$\mathcal{B}_a = \begin{pmatrix} 0 & 0 \\ 0 & -a \end{pmatrix}.$$

This operator is bounded on  $\mathcal{H}$  whenever  $a \in L^\infty(\Omega)$ . Therefore, bearing in mind that  $\mathcal{A} + \mathcal{B}_a$  generates a continuous semigroup, the IBVP

$$(5.3) \quad \begin{cases} \partial_t^2 u + \partial_x^4 u + a(x) \partial_t u = 0 & \text{in } Q, \\ u(0, \cdot) = u(1, \cdot) = 0 & \text{on } (0, \tau), \\ \partial_x u(0, \cdot) = \partial_x u(1, \cdot) = 0 & \text{on } (0, \tau), \\ u(\cdot, 0) = u_0, \partial_t u(\cdot, 0) = u_1 \end{cases}$$

has a unique solution  $u = u_a(u_0, u_1)$  satisfying  $(u, u') \in C([0, T], \mathcal{H}_1) \cap C^1([0, T], \mathcal{H})$ , for any  $\begin{pmatrix} u_0 \\ u_1 \end{pmatrix} \in \mathcal{H}_1$ .

Moreover, the same perturbation argument used in the proof of Theorem 2.2 enables us to show that  $(\mathcal{A} + \mathcal{B}_a, C)$  is exactly observable with constant  $\tilde{\kappa}^2 \geq \kappa^2/2$  provided the norm of the operator  $\mathcal{B}_a$  is sufficiently small. That is, there is  $\delta > 0$  such that for any  $\mathcal{B}_a \in \mathcal{B}(\mathcal{H})$  with  $\|\mathcal{B}_a\| \leq \delta$ , we have

$$(5.4) \quad (1/2)\kappa^2(\|u_0\|_{1/2}^2 + \|u_1\|_0^2) \leq \|\partial_x^2 u(0, \cdot)\|_{L^2((0, \tau))}^2.$$

In light of [21, Lemma 6.10.2, page 218], the spectrum of  $\mathcal{A}$  consists in a sequence of simple eigenvalues  $(i\rho_k)_{k \in \mathbb{Z}^*}$ , where

$$\rho_k = \pi^2 \left( k - \frac{1}{2} \right)^2 + a_k, \quad k \in \mathbb{N}^*,$$

$(a_k)$  a sequence converging exponentially to 0, and  $\rho_{-k} = -\rho_k$ ,  $k \in \mathbb{N}^*$ .

Let  $A_0$  be the unbounded operator on  $L^2(\Omega)$  defined by  $A_0 = \frac{d^4}{dx^4}$  and  $D(A_0) = H^4(\Omega) \cap H_0^2(\Omega)$ . Then  $A_0$  is diagonalizable with eigenvalues  $(\rho_k^2)_{k \in \mathbb{N}^*}$ . Let  $(f_k)_{k \in \mathbb{N}^*}$  be a basis of eigenfunctions, each  $f_n$  is an eigenfunction corresponding to  $\rho_k^2$ . Let

$$g_k = \frac{1}{\sqrt{2}} \left( \frac{f_k}{i\rho_k} \right), \quad \text{and} \quad g_{-k} = -g_k, \quad k \in \mathbb{N}^*.$$

With the help of [21, Lemma 3.7.7, page 101], we get that  $(g_k)_{k \in \mathbb{Z}^*}$  is an orthonormal basis of  $\mathcal{A}_0$ .

Define  $\mathcal{H}_\pm$  as the closure of  $\text{span}\{g_{\pm k}; k \in \mathbb{N}^*\}$ . Then  $\mathcal{H} = \mathcal{H}_- \oplus \mathcal{H}_+$  and  $\mathcal{H}_\pm$  is invariant under  $\mathcal{A}_0$ . We consider  $\mathcal{A}_0^\pm : \mathcal{H}_\pm \rightarrow \mathcal{H}_\pm$  the unbounded operator given by  $\mathcal{A}_0^\pm = \mathcal{A}_0|_{\mathcal{H}_\pm}$  and

$$D(\mathcal{A}_0^\pm) = \{u \in \mathcal{H}_\pm; \sum_{k \in \mathbb{N}^*} k^2 |\langle u, g_{\pm k} \rangle|^2 < \infty\},$$

where  $\langle \cdot, \cdot \rangle$  is the scalar product in  $\mathcal{H}$ , and we set  $\mathcal{A}_{a_0}^\pm = \mathcal{A}_0^\pm + \mathcal{B}_{a_0}$ .

Since  $\rho_{k+1} - \rho_k \rightarrow +\infty$  as  $k \rightarrow +\infty$ ,  $(\rho_k)_{k \in \mathbb{N}^*}$  satisfies the a gap condition. Precisely, there exists  $d > 0$  so that

$$\rho_{k+1} - \rho_k \geq d, \quad k \in \mathbb{N}^*.$$

Set

$$\alpha' = \frac{d}{2\sqrt{2(1+\varrho)}},$$

where  $\varrho$  is as in Section 4.

We have similarly to Theorem 4.1,

**Theorem 5.1.** *Under the assumption*

$$\rho := \|a_0\|_\infty < \alpha',$$

*the spectrum of  $\pm \mathcal{A}_{a_0}^\pm$  consists in a sequence  $(i\mu_k^\pm)$  such that, for any  $\delta \in (0, 1 - \rho^2/(\alpha')^2)$ , there is an integer  $\tilde{k}$  such that*

$$|i\mu_k^\pm - i\rho_k| \leq \bar{\alpha} = \bar{\alpha}(a_0) := \frac{\rho d}{\sqrt{4\rho^2 + d^2\delta}}, \quad k \geq \tilde{k}.$$

*In addition,  $\mathcal{H}^\pm$  admits a Riesz basis  $(\phi_k^\pm) = \left( \begin{pmatrix} \varphi_k^\pm \\ i\mu_k^\pm \varphi_k^\pm \end{pmatrix} \right)$ , each  $\phi_k^\pm$  is an eigenfunction corresponding to  $i\mu_k^\pm$ .*

We define the IB operator  $\tilde{\Lambda}_a$  by

$$\tilde{\Lambda}_a : u_1 \in H_{1/2} \longrightarrow \partial_x^2 u_a(0, u_1)(0, \cdot) \in L^2(0, \tau).$$

One can prove in a straightforward manner that  $\tilde{\Lambda}_a$  is bounded operator between  $\mathcal{H}_1$  and  $H^1((0, \tau))$  and its norm can be uniformly bounded, with respect to  $a$ , by a constant, provided that the  $L^\infty$ -norm of  $a$  is sufficiently small.

We carry out a similar analysis to that after Theorem 4.1 to get the following stability estimate.

**Theorem 5.2.** *Given  $m > 0$ , there exist constants  $C > 0$  and  $\delta > 0$  so that*

$$\|a - a_0\|_0 \leq C \left| \ln \left( C^{-1} \|\tilde{\Lambda}_a - \tilde{\Lambda}_{a_0}\| \right) \right|^{-1/4},$$

if  $a \in a_0 + \delta B_{1,\infty}$  and

$$\sum_{|k| \geq 1} \lambda_k |(a - a_0, \psi_k)|^2 \leq m,$$

where  $\psi_{\pm k} = i\mu_k^{\pm} \varphi_k^{\pm}$ ,  $k \in \mathbb{N}^*$ .

We mention that the method used in this section and in the previous one is easily adaptable to a Schrödinger equation.

## 6. THE CASE OF A HEAT EQUATION

We consider the following IBVP for the heat equation

$$(6.1) \quad \begin{cases} \partial_t u - \Delta u + q(x)u = 0 & \text{in } Q, \\ u = 0 & \text{on } \Sigma, \\ u(\cdot, 0) = u_0. \end{cases}$$

Let  $H^{2,1}(Q) = L^2((0, \tau), H^2(\Omega)) \cap H^1((0, \tau), L^2(\Omega))$ . From [11, Theorem 1.43, page 27], for any  $q \in L^\infty(\Omega)$  and  $u_0 \in H_0^1(\Omega)$ , the IBVP has a unique solution  $u_q = u_q(u_0) \in H^{2,1}(Q)$  and, for any  $m > 0$ ,

$$\|u_q\|_{H^{2,1}(Q)} \leq C \|u_0\|_{1,2},$$

where the constant  $C = C(M)$  is independent on  $q$ ,  $\|q\|_\infty \leq m$ .

Let  $\Gamma$  be an arbitrary nonempty open subset of  $\partial\Omega$  and set  $\Upsilon = \Gamma \times (0, \tau)$ . Using the trace theorem in [11, page 26], we obtain that the following IB mapping

$$\Lambda_q : u_0 \in H_0^1(\Omega) \longrightarrow \partial_\nu u_q(u_0) \in L^2(\Upsilon)$$

is bounded.

The following lemma will be useful in the sequel. Its proof is sketched in Appendix A.

**Lemma 6.1.** *Let  $q_0, q \in L^\infty(\Omega)$  so that  $q \in q_0 + W^{1,\infty}(\Omega)$ . Then  $\Lambda_q - \Lambda_{q_0}$  defines a bounded operator from  $H_0^1(\Omega)$  into  $H^1((0, \tau); L^2(\Gamma))$ . Additionally, for each  $m > 0$ , there exists  $C > 0$  so that*

$$\|\Lambda_q - \Lambda_{q_0}\| \leq C,$$

for all  $q_0, q \in mB_\infty$ . Here,  $\|\Lambda_q - \Lambda_{q_0}\|$  is the norm of  $\Lambda_q - \Lambda_{q_0}$  in  $\mathcal{B}(H_0^1(\Omega); H^1((0, \tau); L^2(\Gamma)))$ .

In the sequel  $\Lambda_q - \Lambda_{q_0}$  is considered as an operator acting from  $H_0^1(\Omega)$  into  $H^1((0, \tau); L^2(\Gamma))$ .

We assume, without loss of generality, that  $q \geq 0$ . Indeed, substituting  $u$  by  $ue^{-\|q\|_\infty t}$ , we see that  $q$  in (6.1) is changed to  $q + \|q\|_\infty$ . So, we fix  $q_0 \in L^\infty(\Omega)$  satisfying  $0 \leq q_0$  and we let  $0 < \lambda_1 < \lambda_2 \leq \lambda_k \rightarrow +\infty$  be the sequence of eigenvalues, counted according to their multiplicity, of  $-\Delta + q_0$  under Dirichlet boundary condition. An orthonormal basis consisting in the corresponding eigenfunctions is denoted by  $(\varphi_k)$ .

Let  $q \in mB_\infty \cap (q_0 + W^{1,\infty}(\Omega))$ . We pick a positive integer  $k$ . Taking into account that  $u_{q_0}(\varphi_k) = e^{-\lambda_k t} \varphi_k$ , we obtain that  $u = u_q(\varphi_k) - u_{q_0}(\varphi_k)$  is the solution of the IBVP

$$(6.2) \quad \begin{cases} \partial_t u - \Delta u + q(x)u = (q_0 - q)e^{-\lambda_k t} \varphi_k & \text{in } Q, \\ u = 0 & \text{on } \Sigma, \\ u(\cdot, 0) = 0. \end{cases}$$

We set  $f = (q - q_0)\varphi_k$  and  $\lambda(t) = e^{-\lambda_k t}$ . Therefore (6.2) becomes

$$(6.3) \quad \begin{cases} \partial_t u - \Delta u + q(x)u = \lambda(t)f(x) & \text{in } Q, \\ u = 0 & \text{on } \Sigma, \\ u(\cdot, 0) = 0. \end{cases}$$

It is straightforward to check that

$$(6.4) \quad u(x, t) = \int_0^t \lambda(t-s)v(x, s),$$

where  $v$  is the solution of the IBVP

$$\begin{cases} \partial_t v - \Delta v + q(x)v = 0 & \text{in } Q, \\ v = 0 & \text{on } \Sigma, \\ v(\cdot, 0) = f. \end{cases}$$

In light of the Carleman estimate in [11, Theorem 3.4, page 165], we can extend [11, Proposition 3.5, page 170] in order to get the following final time observability inequality

$$(6.5) \quad \|v(\cdot, \tau)\|_{H_0^1(\Omega)} \leq C \|\partial_\nu v\|_{L^2(\Upsilon)}.$$

Here  $C$  is a constant depending on  $m$  but not on  $q$ .

By the continuity of the trace operator  $w \in H^{2,1}(Q) \rightarrow \partial_\nu w|_\Upsilon \in L^2(\Upsilon)$ , we get from (6.4)

$$\partial_\nu u(x, t)|_\Upsilon = \int_0^t \lambda(t-s) \partial_\nu v(x, s)|_\Upsilon.$$

We proceed as in the beginning of the proof of Theorem 2.1 to deduce the following estimate

$$(6.6) \quad \|\partial_\nu v\|_{L^2(\Upsilon)} \leq \sqrt{2} e^{\tau^2 \lambda_k^2} \|\partial_\nu u\|_{H^1((0, \tau), L^2(\Gamma))}.$$

On the other hand

$$(6.7) \quad v(x, t) = \sum_{\ell \geq 1} e^{-\lambda_\ell t} (f, \varphi_\ell) \varphi_\ell.$$

Hence

$$\|v(\cdot, \tau)\|_2^2 = \sum_{\ell \geq 1} e^{-2\lambda_\ell \tau} |(f, \varphi_\ell)|^2.$$

Arguing as in Section 3, we get, for any  $\lambda \geq \lambda_1$  and  $N = N(\lambda)$  satisfying  $\lambda_N \leq \lambda < \lambda_{N+1}$ ,

$$(6.8) \quad \|f\|_2^2 \leq e^{2\lambda_k \tau} \|v(\cdot, \tau)\|_2 + \frac{1}{\lambda^2} \sum_{\ell > N} \lambda_\ell^2 |(f, \varphi_\ell)|^2.$$

By Green's formula, we obtain

$$\lambda_\ell (f, \varphi_\ell) = - \int_\Omega \Delta(q - q_0) \varphi_\ell \varphi_k dx + 2 \int_\Omega \nabla(q - q_0) \cdot \nabla \varphi_k \varphi_\ell dx + \lambda_k (f, \varphi_\ell).$$

Therefore, under the assumption that  $q \in q_0 + W^{2,\infty}(\Omega)$  and  $\|q - q_0\|_{2,\infty} \leq m$ ,

$$\lambda_\ell |(f, \varphi_\ell)| \leq (1 + \sqrt{\lambda_k})m + \lambda_k |(f, \varphi_\ell)|.$$

This estimate in (6.8) yields

$$\begin{aligned} \|f\|_2^2 &\leq e^{2\lambda\tau} \|v(\cdot, \tau)\|_2^2 + \frac{2(1 + \lambda_k)m^2 + \lambda_k^2}{\lambda^2} \sum_{\ell > N} |(f, \varphi_\ell)|^2 \\ &\leq e^{2\lambda\tau} \|v(\cdot, \tau)\|_2^2 + \frac{2(1 + \lambda_k)m^2 + \lambda_k^2}{\lambda^2} \|f\|_2^2 \\ &\leq e^{2\lambda\tau} \|v(\cdot, \tau)\|_2^2 + \frac{C\lambda_k^2}{\lambda^2} \|f\|_2^2 \\ &\leq e^{2\lambda\tau} \|v(\cdot, \tau)\|_2^2 + \frac{C\lambda_k^2}{\lambda^2}. \end{aligned}$$

This inequality together with (6.6) imply

$$\|(q - q_0)\varphi_k\|_2^2 = \|f\|_2^2 \leq \sqrt{2} e^{2\tau^2 \lambda_k^2 + 2\lambda\tau} \|\partial_\nu u\|_{H^1((0, \tau), L^2(\Gamma))}^2 + \frac{C\lambda_k^2}{\lambda^2}.$$

But

$$\|\partial_\nu u\|_{H^1((0, \tau), L^2(\Gamma))} = \|\Lambda_q(\varphi) - \Lambda_{q_0}(\varphi_k)\|_{H^1((0, \tau), L^2(\Gamma))} \leq \|\Lambda_q - \Lambda_{q_0}\| \|\varphi_k\|_{H_0^1(\Omega)} \leq \sqrt{\lambda_k} \|\Lambda_q - \Lambda_{q_0}\|.$$

Whence,

$$(6.9) \quad \|(q - q_0)\varphi_k\|_2^2 = \|f\|_2^2 \leq \sqrt{2}\lambda_k e^{2\tau^2\lambda_k^2 + 2\lambda\tau} \|\Lambda_q - \Lambda_{q_0}\|^2 + \frac{C\lambda_k^2}{\lambda^2}.$$

Now the usual way consists in minimizing, with respect to  $\lambda$ , the right hand side of the inequality above. By a straightforward computation, one can see that the minimization argument is possible only if

$$\frac{\lambda_k e^{-2\tau^2\lambda_k^2}}{\|\Lambda_q - \Lambda_{q_0}\|^2} \gg 1.$$

But this estimate does not guarantee that  $\|\Lambda_q - \Lambda_{q_0}\|$  can be chosen arbitrarily small uniformly in  $k$ . However, the minimization argument works if we perturb  $q_0$  by a finite dimensional subspace. That what we will discuss now.

Let  $I > 0$  be a given integer and  $E_I = \text{span}\{\varphi_1, \dots, \varphi_I\}$ . Since  $|(q - q_0, \varphi_k)|^2 \leq |\Omega| \|(q - q_0)\varphi_k\|_2^2$  by Cauchy-Schwarz's inequality, we get from (6.9)

$$\|q - q_0\|_2^2 = \sum_{k=1}^I |(q - q_0, \varphi_k)|^2 \leq C_I \left( e^{2\lambda\tau} \|\Lambda_q - \Lambda_{q_0}\| + \frac{1}{\lambda^2} \right),$$

for some constant  $C_I$  depending on  $I$ . We observe that, according to the preceding analysis,  $C_I$  surely blows-up when  $I \rightarrow +\infty$ .

Minimizing with respect to  $\lambda$  the right hand side of the inequality above, we get

$$\|q - q_0\|_2 \leq C_I |\ln(\|\Lambda_q - \Lambda_{q_0}\|)|^{-1},$$

provided that  $\|\Lambda_q - \Lambda_{q_0}\|$  is sufficiently small. By a simple continuity argument, we see that  $\|\Lambda_q - \Lambda_{q_0}\|$  is small whenever  $\|q - q_0\|_{1,\infty}$  is small. If  $\Lambda_q^I = \Lambda_{q|E_I}$ , we end up getting

**Theorem 6.1.** *Under the preceding notations and assumptions, there exist two constants  $C_I$  and  $\delta_I$  so that*

$$\|q - q_0\|_2 \leq C_I |\ln(\|\Lambda_q^I - \Lambda_{q_0}^I\|)|^{-1},$$

if  $\|q - q_0\|_{1,\infty} \leq \delta_I$ .

*Remark 6.1.* We consider on  $L^2(\Omega)$  the following norm, which weaker than its natural norm,

$$\|w\|_* = \left( \sum_{k \geq 1} e^{-3\tau^2\lambda_k^2} |(w, \varphi_k)|^2 \right)^{1/2}, \quad w \in L^2(\Omega).$$

Then (6.9) yields

$$\|q - q_0\|_*^2 \leq \sqrt{2}e^{2\tau\lambda} \|\Lambda_q - \Lambda_{q_0}\|^2 + \frac{C}{\lambda^2}.$$

We get by minimizing the right hand side with respect to  $\lambda$

$$\|q - q_0\|_* \leq C |\ln(\|\Lambda_q - \Lambda_{q_0}\|)|^{-1}.$$

#### APPENDIX A.

*Proof of Lemma 6.1.* We start by considering the following homogenous IBVP for the heat equation

$$(A.1) \quad \begin{cases} \partial_t u - \Delta u + q(x)u = \phi & \text{in } Q, \\ u = 0 & \text{on } \Sigma, \\ u(\cdot, 0) = \psi. \end{cases}$$

Let  $\phi \in L^2(Q)$  and  $\psi \in H_0^1(\Omega)$ . We obtain by applying one more time [11, Theorem 1.43, page 27] that the IBVP (A.1) has a unique solution  $u_{\phi,\psi} \in H^{2,1}(Q)$  provided that  $q \in L^\infty(\Omega)$ . Moreover, for any  $m > 0$ , there exists  $C > 0$  so that

$$(A.2) \quad \|u_{\phi,\psi}\|_{H^{2,1}(Q)} \leq C (\|\psi\|_{H^1(\Omega)} + \|\phi\|_{L^2(Q)}),$$

uniformly in  $q \in mB_\infty$ .

Let  $A_q$  be the unbounded operator on  $L^2(\Omega)$  defined by

$$A_q = -\Delta + q, \quad D(A_q) = H^2(\Omega) \cap H_0^1(\Omega).$$

Then the solution of (A.1) is given by

$$u_{\phi,\psi}(\cdot, t) = e^{-tA_q}\psi + \int_0^t e^{-sA_q}\phi(\cdot, t-s)ds,$$

where  $e^{-tA_q}$  is the semigroup generated by  $-A_q$ .

In the sequel, we write  $u_\phi$  for  $u_{\phi,0}$ .

Let  $\phi \in C^1([0, \tau]; L^2(\Omega))$  so that  $\phi(\cdot, 0) \in H_0^1(\Omega)$ . Then

$$\partial_t u_\phi(\cdot, t) = e^{-tA_q}\phi(\cdot, 0) + \int_0^t e^{-sA_q}\partial_t \phi(\cdot, t-s)ds.$$

In other words,  $\partial_t u_\phi = u_{\partial_t \phi, \phi(\cdot, 0)}$ . Thus, estimate (A.2) entails

$$(A.3) \quad \|\partial_t u_\phi\|_{H^{2,1}(Q)} \leq C (\|\phi(\cdot, 0)\|_{H^1(\Omega)} + \|\partial_t \phi\|_{L^2(Q)}).$$

Next, let  $\phi \in H^1((0, \tau); L^2(\Omega))$  with  $\phi(\cdot, 0) \in H_0^1(\Omega)$ . Observing that  $u_\phi = u_{\tilde{\phi}} + u_{\phi(\cdot, 0)}$ , where  $\tilde{\phi} = \phi - \phi(\cdot, 0)$ , and  $\partial_t u_{\phi(\cdot, 0)} = u_{0, \phi(\cdot, 0)}$ , we see that is sufficient to consider the case  $\phi(\cdot, 0) = 0$ .

By density, there exist a sequence  $(\phi_k)$  in  $C_0^\infty((0, \tau]; L^2(\Omega))$  converging to  $\phi \in H^1((0, \tau); L^2(\Omega))$ . Armed with (A.2), we get in a straightforward manner that  $u_{\phi_k}$  and  $u_{\partial_t \phi_k}$  converge respectively to  $u_\phi$  and  $u_{\partial_t \phi}$  in  $H^{2,1}(Q)$ . But, in light of the smoothness of  $\phi_k$ ,  $\partial_t u_{\phi_k} = u_{\partial_t \phi_k}$ . Therefore, we have  $\partial_t u_\phi = u_{\partial_t \phi}$  and (A.3) holds true for all  $\phi \in H^1((0, \tau); L^2(\Omega))$  with  $\phi(\cdot, 0) \in H_0^1(\Omega)$ .

Now, let  $q_0, q \in mB_\infty$  so that  $q \in q_0 + W^{1,\infty}(\Omega)$ . Let  $u_0 \in H_0^1(\Omega)$ . By an elementary computation, we get that  $u := u_q(u_0) - u_{q_0}(u_0) = u_\phi$ , with  $\phi = (q_0 - q)u_{q_0}(u_0)$ . Consequently, from the preceding discussion,  $u, \partial_t u \in H^{2,1}(Q)$  and

$$(A.4) \quad \|u\|_{H^{2,1}(Q)} + \|\partial_t u\|_{H^{2,1}(Q)} \leq C \|u_0\|_{H^1(\Omega)}.$$

For some constant  $C = C(m)$ .

Finally, using the continuity of the trace operator  $w \in H^{2,1}(Q) \rightarrow \partial_\nu w \in L^2(\Upsilon)$ , we obtain from (A.4)

$$\|\Lambda_q(u_0) - \Lambda_{q_0}(u_0)\|_{H^1((0, \tau); L^2(\Gamma))} \leq C \|u\|_{H^1(\Omega)}.$$

That is, we proved

$$\|\Lambda_q - \Lambda_{q_0}\| \leq C,$$

where  $\|\Lambda_q - \Lambda_{q_0}\|$  is the norm of  $\Lambda_q - \Lambda_{q_0}$  in  $\mathcal{B}(H_0^1(\Omega), H^1((0, \tau); L^2(\Gamma)))$ .  $\square$

## REFERENCES

- [1] G. ALESSANDRINI, *Examples of instability in inverse boundary-value problems*, Inverse Problems 13 (1997) 887-897.
- [2] C. ALVES, A.-L. SILVESTRE, T. TAKAHASHI AND M. TUCSNAK, *Solving inverse source problems using observability. Applications to the Euler-Bernoulli plate equation*, SIAM J. Control Optim. 48 (2009), 1632-1659.
- [3] M. BELLASSOUED, M. CHOULLI AND M. YAMAMOTO, *Stability estimate for an inverse wave equation and a multidimensional Borg-Levinson theorem*, J. Differ. Equat. 247 (2) (2009), 465-494.
- [4] G. BAO AND K. YUN, *On the stability of an inverse problem for the wave equation*, Inverse Problems 25 (4) (2009), 045003, 7 pp.
- [5] G. BAO AND H. ZHANG, *Sensitivity analysis of an inverse problem for the wave equation with caustics*, J. Amer. Math. Soc. 27 (4) (2014), 953-981.
- [6] M. BELISHEV, *Boundary control in reconstruction of manifolds and metrics (BC method)*, Inverse Problems 13 (1997), R1-R45.
- [7] R. DAUTRAY ET J.-L. LIONS, *Analyse Mathématique et Calcul Numérique*, Vol. VIII, Masson, Paris, 1985.
- [8] C. BARDOS, G. LEBEAU AND J. RAUCH, *Sharp sufficient conditions for the observation, control, and stabilization of waves from the boundary*, SIAM J. Control Optim. 30 (1992), 1024-1065.
- [9] A. L. BUKHGEIM, J. CHENG, V. ISAKOV AND M. YAMAMOTO, *Uniqueness in determining damping coefficients in hyperbolic equations*, Analytic extension formulas and their applications (Fukuoka, 1999/Kyoto, 2000), 27-46, Int. Soc. Anal. Appl. Comput. 9, Kluwer Acad. Publ., Dordrecht, 2001.
- [10] F. CARDOSO AND R. MENDOZA, *On the hyperbolic Dirichlet to Neumann functional*, Commun. PDE 21 (1996), 1235-1252.

- [11] M. CHOULLI, *Une introduction aux problèmes inverses elliptiques et paraboliques*, Springer-Verlag, Berlin, 2009.
- [12] A.V. FURSIKOV AND O.Y. IMANUVILOV, *Controllability of evolution equations*, Lecture Notes Series 34, Seoul National University Research Institute of Mathematics, Global Analysis Research Center, Seoul, 1996.
- [13] V. ISAKOV, *An inverse hyperbolic problem with many boundary measurements*, Commun. PDE 16 (1991), 1183–1195.
- [14] V. ISAKOV AND Z. SUN, *Stability estimates for hyperbolic inverse problems with local boundary data*, Inverse Problems 8 (1992), 193–206.
- [15] H. HOCHSTADT, *Integral equations*, Wiley, NY, 1971.
- [16] A. KATCHALOV, Y. KURYLEV AND M. LASSAS *Inverse boundary spectral problems*, Chapman & Hall/CRC, Boca Raton, (2001).
- [17] I. LASIECKA, J.-L. LIONS AND R. TRIGGIANI, *Non homogeneous boundary value problems for second order hyperbolic operators*, J. Math. Pure et Appl. 65 (1986), 149-192.
- [18] RAKESH AND W. SYMES, *Uniqueness for an inverse problems for the wave equation*, Commun. PDE 13 (1988), 87-96.
- [19] A. A. SHKALIKOV, *On the basis property of root vectors of a perturbed self-adjoint operator*, Proceedings of the Steklov Institute of Mathematics, 2010, Vol. 269, pp. 284-298, Pleiades Publishing, Ltd., 2010. Original Russian Text A. A. Shkalikov, 2010, published in Trudy Matematicheskogo Instituta imeni V.A. Steklova, 2010, Vol. 269, pp. 290-303.
- [20] Z. SUN, *On continuous dependence for an inverse initial boundary value problem for the wave equation*, J. Math. Anal. Appl. 150 (1990), 188-204.
- [21] M. TUCSNAK AND G. WEISS, *Observation and control for operator semigroups*. Birkhäuser Advanced Texts, Birkhäuser Verlag, Basel, 2009.

UR ANALYSIS AND CONTROL OF PDE, UR 13ES64, DEPARTMENT OF MATHEMATICS, FACULTY OF SCIENCES OF MONASTIR,  
UNIVERSITY OF MONASTIR, 5019 MONASTIR, TUNISIA  
*E-mail address:* `kais.ammari@fsm.rnu.tn`

INSTITUT ÉLIE CARTAN DE LORRAINE, UMR CNRS 7502, UNIVERSITÉ DE LORRAINE, BOULEVARD DES AIGUILLETES, BP  
70239, 54506 VANDOEUVRE LES NANCY CEDEX - ILE DU SAULCY, 57045 METZ CEDEX 01, FRANCE  
*E-mail address:* `mourad.choulli@univ-lorraine.fr`